

# Time, Topology and Physical Geometry

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## I.

Tim Maudlin's project sets a new standard for fruitful engagement between philosophy, mathematics and physics. It is an honour to be given a chance to comment on it.

I think we can usefully distinguish two different strands within the project: one more conceptual, the other more straightforwardly metaphysical. The conceptual strand has to do with the proper understanding of concepts standardly analysed in topological terms like continuity, connectedness, and boundary. The metaphysical strand is a particular hypothesis about the geometric aspects of the fundamental structure of reality—the facts in virtue of which the facts about physical geometry are what they are.

What I have to say mostly relates to the metaphysical side of things. But I will begin by saying just a little bit about the conceptual strand.

## II.

In topology textbooks, expressions like 'continuous function', 'connected set of points', and 'boundary of a set of points' are generally given stipulative definitions, on a par with the definitions of made-up words like 'Hausdorff' and 'paracompact'. But as Maudlin emphasises, the former expressions can be used to express concepts of which we have enough of an independent grip to make it sensible to wonder whether the topological definitions are extensionally adequate. And he argues quite persuasively that we have no reason to believe there are. The most important and general argument concerns the live epistemic possibility that physical space is discrete, for example by containing only finitely many points: Maudlin claims that the target concepts can have non-trivial application within physical space even if it is discrete, unlike their putative definitions in terms of 'open set'. Maudlin offers an alternative system of definitions, in terms of the notion of a

set of points constituting a **line** (or of a two-place relation over points constituting a **directed line**), which do not suffer from these problems.

The target concepts are somewhat specialised ones; while the ancient Greeks may have had them, they involve a certain amount of idealisation and abstraction from everyday life. For example, the concept of *connectedness* Maudlin is interested in is not obviously the same as the one I employ when I say that my one-volume *Concise OED* is spatially connected, whereas my two-volume *Shorter OED* is not: that judgment is (arguably) consistent with the claim that even the one-volume book, when looked at at small scales, has the same kind of geometric profile as a swarm of bees. To some extent, then, the project is one of ‘conceptual synthesis’ rather than conceptual analysis: on this way of thinking about it, the question is which idealised, but not arbitrarily specialised, concepts there are in the vicinity of our everyday geometric notions.

On the whole, I find Maudlin’s conceptual claims convincing, which is why I won’t have much more to say about them. My main concern about this aspect of the project involves what Maudlin says about the concept of a *space*. According to Maudlin, ‘[w]e conceive of geometrical spaces primarily by means of... lines’ (p. 67). His view seems to be that for some collection of physical entities to constitute a space, it must be possible, in a non-arbitrary way, to distinguish certain sets of them as **lines** satisfying the axioms for a Linear Structure.

I think this may be too demanding. Consider the following metaphysical hypothesis: there are finitely many “points”, whose structure determines a distinguished function that assigns a non-negative real number  $d(x,y)$  to each pair of points  $x$  and  $y$ . These numbers obey the axioms for a metric space:  $d(x,y) = 0$  iff  $x=y$ ;  $d(x,y) = d(y,x)$ ;  $d(x,z) \leq d(x,y) + d(y,z)$ . And that’s it: nothing about the points determines a non-arbitrary notion of two points being “neighbours”, for example.

This metaphysical hypothesis does have some unattractive features, which emerge when we ask *how it is* that the function  $d$  comes to be “distinguished” from all the other functions from pairs of points to real numbers. Answering this question in a satisfactory way may require adding some new entities to the fundamental ontology over and above

the points, e.g. a collection of new physical entities carrying a structure isomorphic to that of the real line, related to pairs of points by a fundamental ternary ' $d$ -relation'. This is inelegant and uneconomical. Nevertheless, the hypothesis does seem to be at least consistent, and consistent with the existence of creatures with evidence like ours. And to the extent that developments in physics provide reason to question the standard assumption that spacetime is continuous, hypotheses a bit like this should be on our menu of possible alternatives. If we take Maudlin's claim as an a priori or conceptual one, we will have to say that if the hypothesis is true, there is no such thing as space, since there is no non-arbitrary, non-trivial way to assign a Linear Structure to the points. This seems draconian. If we started to gather evidence for the hypothesis, it would be natural to start using geometric expressions like 'distance' to talk about the special function  $d$ . We would be giving up on some intuitive claims about distance, for example that the distance between two points is the length of the shortest line between them. But I doubt that this would amount to a change in the concept.

### III.

On now to the metaphysical strand of Maudlin's project. I take it that one central aim of metaphysics is finding out about the fundamental structure of the world—the actual, physical, concrete world. Of course physics is a vital ally in this inquiry, and given our present ignorance about key questions in physics, we should not expect to be able to make confident pronouncements about any but the most general aspects of the question. Nevertheless, we can make progress in exploring the space of possible hypotheses about the fundamental structure of the world. And in doing so, it makes sense to devote special attention to accounts suggested by actually existing theories in physics, including both fully developed families of mathematically rigorous theories and more speculative suggestions thrown up by current research. We should be especially interested in developing general, flexible hypotheses about fundamental structure that can be filled in in different ways, so as to accommodate a wide range of possible developments in physics.

I take Maudlin's metaphysical proposals in this exploratory spirit. There are really two hypotheses, one more specific than the other. According to the less specific hypotheses, the fundamental entities include *points*, and the fundamental structure over these entities either includes, or very straightforwardly determines, a classification whereby some sets of them count as **lines** satisfying the axioms for a Linear Structure. (Or whereby some sets of ordered pairs of them count as **directed lines** satisfying the axioms of a Directed Linear Structure.) According to the more specific hypothesis, the relevant fundamental structure is a primitive two-place relation among points, which we can pronounce '*x* is earlier than *y*' (or ' $x < y$ ', or '*x* is in the past light cone of *y*', or '*y* is in the future light cone of *x*'); and the **directed lines** are defined in terms of this as intervals of maximal totally ordered subsets of the extension of  $<$ .<sup>1</sup> Maudlin also suggests some additional fundamental structure which could be added to either proposal: a fundamental property of *straightness* instantiated by **lines**, and a fundamental relation of congruence (sameness of length) holding between straight **lines**.

How do these bold metaphysical hypotheses bear on the conceptual side of the project? A strong reading of the conceptual claims would claim that concepts of continuity, connectedness, and boundary—perhaps even the concept of a space—only have nontrivial application if one of the bold metaphysical hypotheses is true. But I doubt that this is what Maudlin intends. Surely he does not regard it as conceptually incoherent to suppose, as some have, that the description of the world in terms of spacetime points and their geometric structure is an emergent structure of some sort, quite far from the fundamental. It is better to think of the conceptual project as having to do with the relations between the *concept* of a line and other intuitive geometric concepts, and as neutral about the question how all these concepts are anchored in fundamental metaphysics.

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<sup>1</sup> Maudlin omits 'intervals of'; but in view of axiom LS<sub>2</sub>, which says that any interval of a **line** is itself a **line**, this must be what he intends.

## IV.

What does it mean to propound a hypothesis about the fundamental structure of the world? According to a standard approach, stating such a hypothesis involves (i) saying something about the *fundamental ontology*—the entities such that all facts ultimately boil down to facts about them; (ii) presenting a *fundamental ideology*, a catalogue of properties of, and relations among, the fundamental entities; and (iii) stating some *laws* which capture important general patterns in the holding of the fundamental properties and relations.

This mode of theorising raises a variety of further questions. Are we supposed to take the fundamental properties and relations seriously as fundamental entities, existing in the same sense as the objects that instantiate them?<sup>2</sup> Are the laws somehow supposed capture the essences of the fundamental properties and relations? Does the claim that the laws are laws add anything important to the claim that they are true? Are there cogent hypotheses about the fundamental structure of the world that cannot be stated in this form, e.g. because they posit fundamental facts which cannot legitimately be thought of as facts about the instantiation of fundamental properties by fundamental objects, and if so, how can these hypotheses be formulated? These are hard questions: fortunately, we don't have to answer them to get started on the enterprise of formulating and comparing specific hypotheses inspired by theories in physics.

One worrying question that is less easily set aside is whether some apparent differences between such hypotheses are merely verbal. Suppose that, in spite of Maudlin's advice, we took seriously the hypothesis that topological *openness* belongs on the list of fundamental properties. Do we really have to choose between this and the hypothesis that

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<sup>2</sup> In Dorr 2007 (§4) I describe two ways of thinking about these questions, 'physical nominalism' and 'structural nominalism', neither of which posits properties and relations as fundamental entities.

takes topological *closedness* as fundamental, defining ‘open set’ as ‘complement of a closed set’ rather than vice-versa? It is hard to accept that there could be a genuine issue here.<sup>3</sup>

But this kind of worry is corrosive. Having got the idea, one will naturally start to wonder whether even superficially very different pictures of fundamental structure might not be mere notational variants. One will be tempted to look for some general principle according to which when there is a natural “translation” between two such theories, or a natural “isomorphism” between the sets of possibilities they leave open, then there is no genuine difference between them. But short of verificationism, there is no known way of formulating such a criterion: it is just too obscure how understand this talk of natural translations and isomorphisms in such a way that it doesn’t just beg the question whether the theories in question are genuine alternatives.<sup>4</sup>

This doesn’t mean that the concerns in question are never warranted. But it does suggest that we should bracket them if we want to get on with the inquiry: until we figure out some general principles for evaluating such claims, there is no point in opportunistically pronouncing that certain differences are merely notational whenever we find it especially hard to take them seriously. No matter how gripped we are by these worries, once we have a particular hypothesis about the fundamental structure of the world on the table, we should search around for variants of it that seem simpler, more economical, or explanatorily more virtuous in some other respect. If the variants are genuinely and not merely verbally different, well and good; if not, then their existence still matters for the purposes of seeing how considerations like simplicity and economy bear on the really genuine questions about fundamental structure, whatever they are.

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<sup>3</sup> Sider (2010) defends a view on which there are genuine, nonverbal questions about which of two properties are fundamental even in cases like this. But Sider’s way of thinking is hard to live with, especially given that it also requires there to be a legitimate question whether ‘and’ or ‘or’ or both or neither is fundamental.

<sup>4</sup> The problem is clearest when the hypotheses we are dealing with happen to include very strong laws—strong enough to pin down the extensions of the fundamental properties and relations uniquely (up to isomorphism). For surely not any two such metaphysical hypotheses are notational variants of one another!

## V.

The general question of fundamental metaphysics is ‘what are the fundamental facts in virtue of which the world is the way it is?’; an important special case is ‘what are the fundamental facts in virtue of which the geometric facts are what they are?’ One class of geometrical facts that especially cries out for explanation are facts about geometrical relations between physical objects and mathematical entities: for example, facts of the form *the ratio of the volume of region A to that of region B is real number x*. Explanations have to stop somewhere, of course; but there is something repugnant about the hypothesis—what Field (1984) calls ‘Heavy Duty Platonism’—that such mixed mathematico-physical relations are fundamental.

There are various philosophies of mathematics that would license a general demand to explain mixed relations in terms of relations among concreta: nominalism, logicism, certain kinds of mathematical structuralism. Even the rather orthodox idea that all mathematical entities are to be identified with sets licenses the demand to some extent: for among the pure sets, there are various equally good candidates to be the set of real numbers, and it seems silly to suppose that the fundamental physical relations privilege one of these (e.g. Dedekind cuts of rationals considered as Wiener-Kuratowski pairs of von Neumann numbers). But whatever one’s views about mathematical ontology might be, it is important to explore ways in which mixed relations might be reduced to fundamental relations between physical objects. For whatever else it might be, mathematics is a useful representational tool; the fact that it is useful to theorise about the physical world by describing its relations to mathematical entities is not much of a reason to assume that these relations are metaphysically fundamental.

Relations between physical objects and real numbers are not the only mixed geometric relations that occur undefined in standard theories in physics. Such theories are often expressed in the language of differential geometry; and the most common approach this subject simply helps itself to the idea that some co-ordinate systems (functions from sets of points to  $n$ -tuples of real numbers) are ‘legitimate’ or ‘admissible’. The intuitive picture is that the admissibility of a co-ordinate system consists in its being faithful to certain as-

pects of the intrinsic structure of the space, but the mathematics is silent about what this intrinsic structure consists in.

Maudlin is generally sympathetic to the project of analysing geometrical relations between the physical and mathematical realms in terms of relations intrinsic to the physical realm. For example, he likes the idea of analysing lengths in terms of some intrinsic notion of ‘congruence’; and he seems appropriately repulsed by the idea that notions like ‘admissible co-ordinate system’ should be fundamental. Nevertheless, his opposition to Heavy Duty Platonism is not total. Maudlin’s **lines** are *sets* of points: and while in the Relativistic part of the proposal the notion of a **line** gets analysed in terms of an ‘earlier than’ relation whose relata are just points, the fundamental property of straightness, and the fundamental relation of congruence, still seem to be instantiated by **lines**.<sup>5</sup>

One way for opponents of Heavy Duty Platonism to build on Maudlin’s work would be to look for further relations just among points in terms of which these affine and metrical properties of **lines** could be defined. A very different approach, which strikes me as more promising, would take **lines** themselves to be concrete physical entities, every bit as real and fundamental as the points.<sup>6</sup>

The latter approach in turn comes in two versions. On the first version, we have something like classical mereology as part of our account of fundamental reality: **linehood** is a fundamental property instantiated by just some of the many mereological fusions of points.<sup>7</sup> On the second version, the fundamental entities are just the points and the **lines**. We have a fundamental relation of ‘incidence’ between points and lines: to be a **line** is just to be something upon which something is incident.

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<sup>5</sup> Maudlin’s main reason for wanting this seems to be the desire that facts about long-range relations among points, like distance, should turn out to be extrinsic and hence non-fundamental. As I said in §II, I am not convinced that this is a good a priori constraint to put on our theorising.

<sup>6</sup> One could make do with just the points if one were willing to countenance fundamental ‘plural properties’ or ‘multigrade relations’: then linehood could be thought of as such a property, instantiated by many points at a time. Whether this is legitimate is a deep foundational question.

<sup>7</sup> If we wanted **directed lines**, we would need something more complicated—entities that stand to fusions of points as sets of ordered pairs of points stand to sets of points.

Some hold on a priori grounds that the true catalogue of fundamental relations will include a relation of *parthood* subject to the laws of classical mereology. These people will of course be drawn to the first version of the approach: if fusions of points are going to be in the ontology anyway, it seems more economical to identify the **lines** with some of these fusions, and identify incidence with parthood, rather than having **lines** as an additional supply of mereological atoms. But I don't think that there are good a priori grounds to expect the fundamental structure of the world to include anything like mereological structure. Granted, if it doesn't, it may be hard to find entities in the fundamental ontology which we could plausibly identify with ordinary objects like chairs, tables, planets and people. But is there any reason to think that there are any such things, in the sense of 'there are' relevant to fundamental ontology? In my view, the sense of 'there are' in which it is obvious that there are chairs, tables, planets and people is something different.<sup>8</sup>

## VI.

Once we have set aside our temptations to play the 'mere notational variant' card, we should be prepared to find that, even after we have settled on a general strategy like 'Take lines as fundamental!', there are many slightly different ways to implement the strategy in a hypothesis about fundamental structure. Once we have a particular hypothesis on the table, we can start tinkering with it to see if we can simplify its ontology or its ideology.

Maudlin's account of Relativistic spacetime embodies one such simplification. We might have thought that capturing this the geometric structure would require a rich Linear Structure, with spacelike, timelike and lightlike **lines**, and mixed **lines** made up of different kinds of segments. But as Maudlin shows, this is needlessly uneconomical: we can throw away all but the timelike-or-lightlike **lines** and still recover facts about the distances between spacelike-separated points, by defining them in terms of facts about the lengths of timelike-or-lightlike **lines**. (And once we have done this we can define **linehood** itself in terms of 'earlier than', provided we don't mind excluding spacetimes with closed timelike curves.)

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<sup>8</sup> See Dorr 2007, §1.

Can we find other simplifications of a similar sort? Here is one idea: if we are eventually going to need a fundamental property of *straightness* that distinguishes a special sub-structure of **lines**, why not simplify the ideology and ontology by throwing away all the non-straight **lines**? In a relativistic spacetime, a specification of the straight timelike/lightlike **directed lines** and their congruence relation should be enough to pin down up to isomorphism (and a scale factor) the geometric structures required by standard presentations of the physics. Given the fundamental structure of straight **lines**, we could define up derivative Directed Linear Structures containing non-straight lines—for example, we can build co-ordinate systems whose co-ordinate curves are straight **lines**, and define **lines** in a new broader sense as graphs of a curve whose co-ordinate functions are all **continuous**.

The opening pages of Maudlin’s paper suggest that he would not be sympathetic to the idea that “rubber sheet geometry” is in this way derivative from affine geometry (straightness structure). He writes that

The affine structure itself presupposes an even more basic organization of the points. The straight lines in a space are a subclass of the continuous curves, and the continuous curves are defined, mathematically, independently of the affine structure. So sitting at the bottom of this definitional hierarchy is a sub-metrical geometry, aspects of a space that do not depend on either the metric or the affinity.

It seems to me that the mathematical sense of ‘fundamental’, in which topology is said to be ‘more fundamental’ than affine and metric geometry is quite different from the metaphysical sense of ‘fundamental’ we are concerned with. The mathematical fundamentality of topology is a kind of generality: there are many kinds of mathematical structure within which there is a natural way of defining a property of sets of points which obeys the topological axioms for ‘open set’; this makes topology useful for capturing behaviour common to many mathematical structures. There is nothing in this to rule out the hypothesis that the metaphysically fundamental facts about physical space are all facts about its affine or metric structure.

There are other kinds of potential simplifications we can consider once we start tinkering with the basic picture. For example, if we are going to have **lines** in the fundamental ontology in any case, we might consider simplifying the ontology by getting rid of points

as fundamental entities, and doing everything with some fundamental relations among **lines**.

One strategy would take as fundamental the relation of two lines “overlapping”—intuitively, sharing some point in common. In terms of this, we can define what it is for two lines to *cross*, or “share exactly one point in common”:  $\lambda_1$  crosses  $\lambda_2$  iff  $\lambda_1$  overlaps  $\lambda_2$ , and there are lines  $\lambda_3$  and  $\lambda_4$  neither of which overlap  $\lambda_2$  such that both  $\lambda_3$  and  $\lambda_4$  are parts of  $\lambda_1$  and every line that is part of  $\lambda_1$  overlaps either  $\lambda_3$  or  $\lambda_4$ . (One line is *part* of another if every line that overlaps the former overlaps the latter.) If we are dealing with **directed lines**, we will want, instead, some fundamental relation like ‘ $\lambda_1$  overlaps  $\lambda_2$  later than  $\lambda_3$  does’: we think of **directed lines** as determining an order (strictly, a total preorder) among the lines they cross, rather than an order among points. Given this fundamental structure, we should be able to code up points as equivalence classes of pairs of crossing lines.<sup>9</sup>

This reconstruction will break down in some very simple Linear Structures. For example, there is the trivial Linear Structure consisting of a single, two-point line: obviously we cannot identify these points with two different equivalence classes of pairs of lines. But I don’t think this is much of a problem: we should not dismiss a hypothesis about fundamental structure just because it admits a narrower range of possibilities than another account, especially if the possibilities in question are ones that we have no empirical reason to take seriously.<sup>10</sup>

Another approach would be to take parthood as primitive rather than defining it in terms of overlap. Indeed, we might not need anything else. I haven’t got any neat proofs, but it seems likely that there is some large and interesting class of Linear Structures within

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<sup>9</sup> To carry out this construction, we need to be able to say that  $\lambda_1$  and  $\lambda_2$  cross at the same point where  $\lambda_3$  and  $\lambda_4$  cross. There is always the option of taking this as a further fundamental relation. But at least in some well-behaved Linear Structures, it should be definable in terms of overlap. First, an obvious extension of the definition of *crossing* lets us define what it is for two lines to cross a third at the same point. And it seems intuitive that when  $\lambda_1$  and  $\lambda_2$  cross at the same point where  $\lambda_3$  and  $\lambda_4$  cross, we can find a line that crosses  $\lambda_1$  and at least one of  $\lambda_3$  or  $\lambda_4$  at that point. (The only cases I can think of where this fails involve quite bizarre and degenerate Linear Structures.) If this holds, we can define ‘ $\lambda_1$  and  $\lambda_2$  cross at the same point where  $\lambda_3$  and  $\lambda_4$  cross’ as ‘for some  $\lambda_5$  that crosses  $\lambda_1$  where  $\lambda_2$  does, either  $\lambda_5$  crosses  $\lambda_3$  where  $\lambda_4$  does and  $\lambda_1$  crosses  $\lambda_5$  where  $\lambda_3$  does, or  $\lambda_5$  crosses  $\lambda_4$  where  $\lambda_3$  does and  $\lambda_1$  crosses  $\lambda_5$  where  $\lambda_4$  does.’

<sup>10</sup> Cf. Maudlin on closed timelike curves.

which the facts about the subset relation among lines pins down the whole structure up to isomorphism. (However, there are other physically interesting Linear Structures in which this is not so, e.g. the Segment-Spliced Linear Structure of straight lines in Euclidean space.) Such a reduction of geometrical notions to mereological ones will be especially interesting to those (see §V above) who think they have *a priori* reason to include *parthood* on the list of fundamental relations in any case. However, as a matter of sociology, most of those who hold this view will also accept the mereological axiom of universal composition on *a priori* grounds, in which case their ontology will have to include fusions of **lines** that are not themselves **lines**, and their ideology will thus need a fundamental property to differentiate the **lines** from the non-**lines**.

## VII.

When we are trying to figure out how to divide our credence in a reasonable way between different hypotheses about fundamental structure, considerations of simplicity will matter a lot. What we want is not just a short list of fundamental properties and relations, but a simple-yet-strong set of laws stated in terms of these properties and relations, in terms of which we can give satisfactory explanations of a wide range of evidence.

Making these discriminations will require a well-honed sense of what makes for “explanatory satisfactoriness”. One important way in which laws can fail to be satisfactory is for them to take an ‘as if’ form. Someone who wanted to admit atoms but not subatomic particles into their fundamental ontology could write down a law of the form ‘the motions of atoms are just as they would be if they were composed of subatomic particles obeying such-and-such laws’. Although such laws need not be especially complex in any obvious sense of ‘complex’, they are obviously explanatorily unsatisfactory: the question ‘*why* do the atoms move around like that’ cries out for an answer. (See Dorr 2010, §4.) Finding laws which avoid this kind of badness can be hard task. Suppose we are trying to write down a complete set of laws for some special-relativistic physics, as part of which we want to describe the structure of a Minkowski spacetime in terms of Maudlin’s fundamental relations. It is not enough just to say that ‘earlier than’ is a partial order, or that the in-

tervals of maximal totally-ordered subsets form a Directed Linear Structure. This is far from sufficient to pin down the structure of Minkowski spacetime: if our laws said no more than this, they would be much too weak to do the necessary explanatory work. One thing we *could* say that would not be too weak is this: the set of spacetime points admits a co-ordinate system such that  $x$  is earlier than  $y$  iff the co-ordinates of  $x$  and  $y$  satisfy the standard co-ordinate definition of one point belonging to the past light cone of another. But this invocation of co-ordinate systems is worryingly reminiscent of the atom-lover's 'as if' law. In effect, we are saying that the facts about the 'earlier than' relation are just *as if* spacetime had all this extra co-ordinate structure related to the 'earlier than' structure in a particular way. It would certainly be much nicer if we could state some laws *directly* in terms of 'earlier than' which entail the existence of appropriate Minkowski co-ordinate systems, in the same way that the intrinsic axiomatisations of Euclidean geometry developed by Hilbert and Tarski entail the existence of Cartesian co-ordinate systems.

Admittedly, the analogy between the obviously bad 'as if' law and the law that says that there is an admissible co-ordinate system is far from perfect: to properly assess the stringency of the demand for explanatorily satisfactory laws, we will need a more thoroughly worked out account of what the relevant kind of badness consists in.<sup>11</sup> But *prima facie*, it seems to me that in choosing among competing lists of fundamental relations, it matters a lot to see which lists allow us to state strong, satisfactory, "intrinsic" laws, and which require us to resort to suspicious devices like existential quantification over co-ordinate systems.

I don't know how well Maudlin's favoured fundamental relations do by this criterion. I don't know how to write down "intrinsic" laws about these relations strong enough to pin down, say, the distinctive geometric structure of Minkowski spacetime, or of a vacuum solution to general relativity; but that isn't to say it can't be done. Still less do I know how to add additional fundamental relations to describe some sort of physical content in spacetime—say, the electromagnetic field—in such a way that I could state a satisfactory

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<sup>11</sup> I make a start on this in Dorr 2010.

system of laws governing both the geometry and the physics. Figuring these things out is a big task, which may require considerable technical ingenuity.

### VIII.

Theories of fundamental structure based on Maudlin's ideas are attractive. But the right slogan for this stage of our enquiries should be 'Let a thousand flowers bloom'. As part of this horticultural endeavour, we should pay special attention to parts of the garden in which we can take metaphysical inspiration from the mathematics used in actual theories in mathematical physics. For we can hope, by doing this, to find systems in which the task of extracting explanatorily satisfactory laws stated in terms of the fundamental relations from existing theories in mathematical physics will be especially easy.

The standard mathematical apparatus used for stating physical theories about spacetime is that of differential geometry. Everything is done on the assumption that physical spacetime forms a *smooth manifold*: a mathematical structure somewhat richer than a mere topological space, which one can think of as describing the structure of a somewhat less flexible "rubber sheet" which can be deformed at will, but not in such a way as to introduce any "kinks" or "corners". On the most common approach, this means that we are somehow given an 'atlas' of admissible co-ordinate systems. But let me sketch another well-known approach, which I think may be better adapted to metaphysical purposes.<sup>12</sup> On this way of proceeding, what we are given is a privileged class of *smooth functions* (also known as ' $C^\infty$  functions') within the set of all functions from physical points to real numbers, or 'scalar fields'. The functions from physical points to real numbers form a *ring*, in the sense that we can define well-behaved notions of addition ( $(f+g)(p) =_{\text{df}} f(p)+g(p)$ ) and multiplication ( $(fg)(p) =_{\text{df}} f(p)g(p)$ ). The smooth functions are a subring of this ring (a subset

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<sup>12</sup> My main source here is chapter 4 of Penrose and Rindler 1984; thanks to Frank Arntzenius for pointing me towards this. .

closed under addition and multiplication), and the constant functions are a subring of the ring of smooth functions.<sup>13</sup>

Armed with these primitive distinctions, one can define further mathematical structures associated with the manifold. A *smooth vector field* on the manifold is a function  $V$  from smooth scalar fields to smooth scalar fields such that (i)  $V(f) = 0$  when  $f$  is a constant function; (ii)  $V(f+g) = V(f)+V(g)$ , and (iii)  $V(fg) = fV(g)+gV(f)$  (the Leibniz product rule). We can define addition on vector fields by  $(V_1+V_2)(f) =_{\text{df}} V_1(f)+V_2(f)$ , and multiplication of smooth vector fields by smooth scalar fields by  $(fV)(g) =_{\text{df}} fV(g)$ . We define a *smooth covector field* on the manifold as a function  $\omega$  from smooth vector fields to smooth scalar fields that is “ $C^\infty$ -linear”, in the sense that  $\omega(V_1+V_2) = \omega(V_1)+\omega(V_2)$  and  $\omega(fV) = f\omega(V)$ . Finally, a smooth tensor field of rank  $m,n$  is a function that takes in  $m$  smooth covector fields and  $n$  smooth vector fields and yields a smooth scalar field, and that is  $C^\infty$ -linear in each argument. In physics, we pick out some of these mathematical entities as (somehow) physically distinguished. In a (false) theory of a fundamental continuous fluid, a vector field is physically distinguished as ‘the velocity field of the fluid’. In electromagnetism, a tensor field of rank 0,2 is physically distinguished as the *electromagnetic field tensor*. One of the physically distinguished tensor fields (also of rank 0,2) that makes an appearance in almost every physical theory is the *metric*. This tensor plays a special relation in the analysis of geometric facts: for example, notions like the length of a path or the volume of a region are standardly defined in terms of the metric. But from the point of view of the physical laws, the metric is just another physically distinguished tensor field: to state the physical laws, we define up various further tensors (e.g. the Ricci tensor, the stress-energy tensor) in terms of the these, and state equations (e.g. Einstein’s field equation for general relativity) involving the results.

If we are exploring alternatives to Heavy Duty Platonism, we will need to find some nontrivial answer to the question *what it is* for a function from spacetime points to real numbers to be smooth, or for a function from pairs of vector fields to scalar fields to be the

<sup>13</sup> Note that there is no need to add topological notions as additional primitive structure over and above this. We can take the topology of the space as defined by taking the open sets to be (unions of) the sets  $\{p: f(p)>0\}$ , where  $f$  is smooth. See Penrose and Rindler 1984, p. 181.

metric. One strategy would be to analyse these mixed properties and relations in terms of relations all of whose relata are *points*; but this looks very challenging. A general moral we can draw from Maudlin's theory of lines is that it helps a lot to have a fundamental ontology that contains some entities besides the points. What could the additional entities be? A conservative approach would identify them as certain mereological fusions of points. But as I have already said, I don't think mereology has any special status when we are doing fundamental ontology. If we want to posit a fundamental relation subject to laws which make it natural to pronounce it 'part of', we must do so on the same a posteriori grounds for which we would posit any other piece of fundamental structure. We should be careful not to overlook alternatives to and generalisations of mereological structure just because of their unfamiliarity. What I want to suggest is that instead of enriching our fundamental ontology with entities which behave like *sets* of points, we should consider enriching our fundamental ontology with entities which behave like *functions from points to real numbers* (scalar fields). I will call these putative entities 'Scalars'. But it is important they are not supposed to be *identical* to scalar fields in the ordinary sense. The latter are mathematical entities: functions from points to real numbers. The Scalars, by contrast, are concrete physical entities whose fundamental relations to points and to one another determine a natural correspondence between them and scalar fields.

Fully filling in this theory will require specifying the fundamental relations which confer this structure on the Scalars. There are various ways to do this. For the sake of definiteness, let's suppose we use (a) ternary 'sum' and 'product' relations relating Scalars; (b) the ternary relation 'Scalars  $s_1$  and  $s_2$  coincide on point  $p$ ', and (c) a fundamental property *constancy* that distinguishes the Scalars corresponding to constant functions. In terms of these relations we can state laws of plenitude which "say" that there is a Scalar corresponding to every function from points to real numbers in the same sense in which the laws of classical mereology "say" that there is a region corresponding to every set of points.

Given this ontology and ideology, it is a relatively straightforward matter to begin writing down fundamental laws based on some physical theory couched in the language

of differential geometry. First we will need to capture the “smooth manifold” structure of spacetime by introducing a new fundamental property *smoothness* which distinguishes a special class of Scalars, subject to some distinctive laws.<sup>14</sup> And then, if the mathematical physics we are trying to recover talks about a distinguished vector field, say the “fluid velocity field”, we can add a corresponding new primitive binary relation over Scalars: “The fluid velocity field maps  $s_1$  to  $s_2$ ’.

Things get trickier when the physics includes distinguished tensor fields other than vector and scalar fields. The obvious approach would be to add two new systems of fundamental entities corresponding to vector fields and covector fields. But this seems ontologically extravagant. Fortunately, it may not be necessary. First of all, we can do without covector fields as fundamental entities. In differential geometry there is a special function  $d$  from scalar fields to covector fields, defined by  $df(V) = V(f)$ . A covector field is said to be *exact* iff it is the result of applying  $d$  to some scalar field. Although not every covector field is exact, each tensor field of rank  $m, n$  is fully determined once we know what scalar field it yields as output given any  $m$  *exact* covector fields and  $n$  vector fields as input. Thus if, for example, we are trying to reconstruct a distinguished tensor field  $T$  of rank  $2, 0$ , we introduce a new fundamental three-place relation over Scalars, corresponding to the mathematical relation  $T(df_1, df_2) = f_3$ .

That takes care of tensors of rank  $m, 0$ , but still leaves us with no way to deal with other tensor fields short of enriching the fundamental ontology with new entities corresponding to vector fields.<sup>15</sup> But there is a trick that we can use to avoid this. As I said above, in the context of physics the set of physically distinguished tensor fields normally includes a special one called the “metric”,  $g$ , of rank  $0, 2$ . Like any tensor field of rank  $0, 2$ ,  $g$  determines a function  $\Phi_g$  from vector fields to covector fields, defined by  $\Phi_g(V_1)(V_2) =_{df} g(V_1, V_2)$ . And in most of the contexts that come up in physics, including that of Relativity theory, the metric is required to be *non-degenerate*, which means that  $\Phi_g$  is a *bijection* be-

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<sup>14</sup> For the mathematical form of these laws, see Penrose and Rindler 1984, pp. 180–82. Antzenius (2010) shows how to state nominalistic versions of the laws.

<sup>15</sup> See Arntzenius 2010 for one way to do this.

tween vector fields and covector fields. The upshot is that we can use the metric to go back and forth as we please between vector fields and covector fields, and thus between tensors of rank  $m, n$  and tensors of rank  $m+n, 0$ . In this friendly context at least, we can reconstruct tensor fields of all sorts from fundamental relations among Scalars: to capture a tensor field  $T$  of rank  $m, n$ , we can use a fundamental  $m+n+1$ -place relation over Scalars corresponding to the mathematical relation  $T(df_1, \dots, df_m, \Phi_g^{-1}(df_{m+1}), \dots, \Phi_g^{-1}(df_{m+n})) = f_{m+n+1}$ .<sup>16</sup>

(I have been ignoring complications induced by the arbitrariness of units. For example, if we don't want to have a metaphysically privileged unit of distance, we shouldn't really want our fundamental relations to pin down the metric tensor *uniquely*—rather, there should be a one-dimensional family of “equally good” candidates to be the metric tensor, each corresponding to a choice of unit. The obvious way to achieve this neutrality involves adding new argument places to the fundamental relations, by analogy with the move from a ‘length’ relation between lines and numbers to a ternary ‘length-ratio’ relation between pairs of lines and numbers. I won't go into the details.)

If I were advertising the ontology of Scalars as a way of vindicating nominalism, you would have a right to be suspicious. In respect of the fundamental relations they instantiate and the characteristic laws which govern those relations, Scalars do not look much like our paradigmatic examples of concrete objects. On the other hand, spacetime points have by now come to be generally classified as ‘concrete’ in spite of the fact that (in many theories) they are governed by laws which make them behave just like certain mathematical entities. For my part, I don't care at all about the labels ‘concrete’ and ‘abstract’. What I care about is finding an economical fundamental ontology and ideology in terms of which I can state strong and simple laws. I don't mind borrowing ideas from mathematics about what this structure might look like, and I am not at all worried that in doing so I will somehow have started down a slippery slope at the end of which is a bloated fundamental ontology containing all of mathematics.

<sup>16</sup> This applies equally to the metric itself, the facts about which will be captured by a fundamental ternary relation among Scalars corresponding to the mathematical relation  $g(\Phi_g^{-1}(df_1), \Phi_g^{-1}(df_2)) = f_3$ , or equivalently,  $\Phi_g^{-1}(df_1)(f_2) = f_3$ .

Besides, Scalars are not really that dissimilar to other putative kinds of entities that have generally been accepted as concrete. Spacetime regions have generally been accepted no less nominalistically kosher than spacetime points. But one way to think of those Scalars which lie between 0 and 1 is as “fuzzy regions”, which points can belong to to different degrees.<sup>17</sup> This doesn’t seem so strange, does it? Admittedly, it is harder to get any such intuitive purchase on the rest of the Scalars. But if we were really worried about this, we could make do with the more restricted set of Scalars—by using some smooth bijection between  $[0,1]$  and the real line, we could treat them as proxies for the full set of scalar fields, refitting the fundamental relations corresponding to physical tensor fields in such a way as to take this representation into account.<sup>18</sup>

## IX.

Here, then, is a possible fundamental structure reality might have: an ontology of points and Scalars, with fundamental operations on Scalars giving them the structure of a ring; a fundamental ‘coincidence at a point’ relation; a three-place relation on Scalars corresponding to the metric tensor; and perhaps some more relations on Scalars corresponding to other physically distinguished vector and tensor fields. This is certainly a less intuitive picture than one based on Maudlin’s **lines**. But I doubt this kind of intuitiveness is very important in fundamental metaphysics, especially if the sacrifice of intuitiveness makes it possible to state intrinsic, explanatorily satisfactory versions of physical laws.

We could consider simplifying both the ontology and the catalogue of fundamental relations in various ways. I will mention four possible simplifications.

(i) We could do without the idea of a privileged unit Scalar. This would mean thinking of various operations on Scalars, including multiplication, as making sense only relative to

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<sup>17</sup> I am using ‘0’ and ‘1’ here as names, respectively for the Scalar which, when added to any other Scalar, yields that same Scalar as result, and the Scalar which when multiplied by any other Scalar, yields that same Scalar as result.

<sup>18</sup> The theory I am sketching has a variant in which Scalars really are just regions—but not regions of a standard four-dimensional spacetime, but regions in a five-dimensional space, whose points work like ordered pairs of old-style points and real numbers. Arntzenius (2010) makes a strong case that in the context of gauge-symmetric field theories, if one is going to believe in points at all, one should believe in the points of some such higher-dimensional space (a fibre bundle).

an arbitrary choice of a given constant Scalar to serve as unit: some of our fundamental relations would then require an extra argument place to capture this unit. In this way we could avoid the need for a fundamental relation corresponding to multiplication of Scalars: for the addition facts together with the facts about which Scalars are constant suffice to fix the extension of the four-place relation ' $f_1 f_2 = f_3$  relative to the choice of  $f_4$  as unit'.

(ii) Instead of a generous ontology with Scalars corresponding to arbitrary functions from points to real numbers and a fundamental property of smoothness picking out some of them as special, we could try to get by with a more economical ontology in which we only have Scalars corresponding to smooth scalar fields in the first place. The cost of this is that it is not obvious how to state a 'plenitude' axiom guaranteeing that there as many Scalars as we want there to be in an explanatorily satisfactory way. It should not be a problem if we don't mind laws which appeal to mathematical ontology, or using higher-order logic of some sort; but evaluating such laws raises difficult issues as discussed in §VII.

(iii) If the topology of the whole spacetime is non-compact (roughly: it is infinite in extent), we might squeeze out a little more ontological economy by restricting ourselves not only to smooth Scalars, but to smooth Scalars which are zero everywhere outside some compact region. This will mean getting rid of constant Scalars: the work previously done by fundamental property of constancy can be done instead by a fundamental relation  $s_1$  is constant wherever  $s_2$  is nonzero.

(iv) If we are going to have Scalars in the ontology in any case, it is natural to ask whether we can make things a bit more unified by getting rid of the separate category of *points*. This is easily done if we have the rich ontology of Scalars corresponding to arbitrary functions. We can 'code up' points as special sets of Scalars: intuitively, those that are zero everywhere except for the given point. The ring structure of the Scalars is enough to tell us which sets of Scalars correspond to points in this way: the relevant sets are those that (a) are closed under addition; (b) are closed under multiplication by any Scalar; (c) contain at least one nonzero Scalar, and (d) have no proper subsets meeting conditions (a-c). (These are called the *minimal ideals* of the ring.) We no longer need a separate funda-

mental relation of ‘coincidence at a point’: we can say that  $s_1$  and  $s_2$  “coincide at the point represented by minimal ideal  $S$ ” iff  $ss_1 = ss_2$  for each  $s \in S$ .

This won’t work if we have adopted simplification (ii), getting rid of all but the smooth Scalars. But a closely related construction still will: we can represent points as *maximal* ideals—sets of scalars that meet conditions (a-c) and are not proper subsets of any other sets that meet these conditions, except for the set of all Scalars. Intuitively, the Scalars in the set are those that are zero at the given point. This will be reflected in our new definition of ‘coincidence at a point’:  $s_1$  and  $s_2$  “take the same value at the point represented by maximal ideal  $S$ ” iff  $s_1 - s_2 \in S$ .

The upshot in either case is that points are superfluous: the ring structure of the Scalars—in fact, merely the addition structure of the Scalars, together with the information about which Scalars are constant—is enough to pin down a smooth manifold up to diffeomorphism.<sup>19</sup>

A fundamental ontology comprising nothing but Scalars is somewhat alien to our ordinary ways of thinking. We are used to associating fundamentality with smallness of size; whereas to the extent that we can sensibly think of Scalars as having sizes, most of them are enormous. The vision is as different as can be from that of Humean Supervenience (Lewis 1986). But this is not such a novelty: theories of ‘gunky spacetime’ (e.g. Arntzenius 2008) propose an ontology in which big and small objects are on a par, and in which points don’t exist at all except as constructions out of regions.

Could there be people like us, having evidence like ours, in a world where the only fundamental things were Scalars? I think it will be hard to maintain that there could not, if one is comfortable with the idea that people are going to turn out to be non-fundamental entities in any case, and with a broadly functionalist picture of properties like personhood. The idea that there could be people in such a world challenges the as-

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<sup>19</sup> Geroch (1972) suggests using the ring of Scalars (an ‘Einstein Algebra’) as an alternative to an ontology containing points, although it is not clear whether he regards it as more than a useful notational variant. Earman (1989) also promotes the approach (‘Leibniz algebras’), although I don’t think it has the particular advantages over the standard ontology which he claims on its behalf.

sumption that non-fundamental entities are “built” out of fundamental entities as walls are built out of bricks. But that is an assumption that needs to be challenged in any case.<sup>20</sup>

There are many other variants of the approach which we could consider, and which we might be led to consider by looking at the details of particular physical theories. For example, it isn’t really crucial that Scalars work like functions from points to real numbers: there are other kinds of spaces of values that would do as well for the purposes of defining vector and tensor fields, and which might be natural candidates to use if we were going to need them anyway for the purposes of physics.<sup>21</sup> Separating the investigation of the metaphysical foundations of physical geometry from the investigation of the metaphysical foundations of physics as a whole works well enough as a simplifying device. But ultimately, we just care about the fundamental structure of the world. Given how intimately geometry is bound up with the rest of physics, it would be foolish to assume there will be any useful way to separate off the geometric aspects of the fundamental structure from the rest.<sup>22</sup>

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<sup>20</sup> I don’t want to give the impression that you need to buy in to a framework of ‘fundamental’ and ‘non-fundamental’ entities (or kinds of quantification) in order to take the ontology of Scalars seriously. We could claim that chairs, people, and so on just *are* certain Scalars—e.g. Scalars that are zero at points intuitively “unoccupied” by the object in question, or Scalars that are nonzero at such points. I don’t see that such an identification is any more problematic than the identification of people with regions of spacetime (Sider 2002, §4.8). There is the issue that there is a vast multiplicity of Scalars which seem equally well qualified to be identified with a given ordinary object; but this is just another instance of the ‘Problem of the Many’, not different in principle from the difficulty in deciding on the exact borders of the region identical to a given ordinary object.

<sup>21</sup> Also, I don’t think it is really crucial that we be able to make sense in an absolute way of comparisons between the values of Scalars at different points: I am hopeful that it would be enough for the Scalars to have the structure of the space of smooth sections of a fibre bundle carrying a *connection* that lets one make sense of a *local* notion of ‘constancy’. (See Maudlin 2007, chapter 3 and Arntzenius 2010 for explanations of these notions.) This would be natural given the role of such fibre bundles in modern physics. But at present I don’t have a good sense of how to state explanatorily satisfactory laws about entities with this less-rich structure.

<sup>22</sup> Thanks to Harvey Brown for spurring me to take this on, to Tim Maudlin for support and inspiration, and especially to Frank Arntzenius, for many hours of discussion without which I wouldn’t have the least clue what to think about these matters.

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