Generality, Extensibility, and Paradox

J.P. Studd
University of Oxford
GENERALITY, EXTENSIBILITY, AND PARADOX

J. P. STUDD
UNIVERSITY OF OXFORD

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CONTACT
mail@aristoteliansociety.org.uk
www.aristoteliansociety.org.uk

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BIOGRAPHY

James Studd is the University Lecturer in the Philosophy of Mathematics at the University of Oxford and a Fellow and Tutor at Lady Margaret Hall. In addition to the philosophy of mathematics, he works on the philosophy of logic, with occasional forays into the philosophy of language and metaphysics. He is currently writing a book about absolute generality (forthcoming with OUP).

EDITORIAL NOTE

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Absolutism is the view that quantifiers like ‘everything’ sometimes range over an absolutely comprehensive domain. The debate between absolutists and the relativists opposing them comes down, in significant part, to a trade off between generality and collectability. But to reach this conclusion we must dispense with some long-standing concerns over the coherence of the argument from the paradoxes in favour of relativism and a heterodox absolutist response that seeks to reconcile the indefinite extensibility of set with the availability of an absolutely comprehensive domain.

I.

WHAT IS THE RELATIONSHIP between Russell’s paradox, indefinite extensibility, and the availability of quantification over absolutely everything?

Let’s provisionally gloss absolutism about quantifiers as the view that sometimes—when we remove any explicit or tacit restrictions—quantifiers like ‘everything’ or \( \forall x \) interpreted according to their standard, truth-conditional semantics, range over an absolutely comprehensive domain: a domain comprising absolutely everything.

Michael Dummett famously contends that the ‘prime lesson’ of the set-theoretic paradoxes is the failure of absolutism (1981, p. 516). More generally, in addition to impugning (classical) quantification over absolutely every object, the paradoxes rule out quantification over absolutely every \( F \), whenever \( F \) is an ‘indefinitely extensible’ concept.\(^1\) This is because, according to Dummett, an indefinitely extensible concept is ‘one such that, if we can form a definite conception of a totality all of whose members fall under that concept, we can, by reference to that totality, characterize a larger totality all of whose members fall under it’ (1994b, p. 22). The concept set provides a paradigm example. Given a domain, we can characterize the domain’s ‘Zermelo-Russell set’, as the set whose elements are the members of the domain that lack themselves as elements. The assumption that the domain contains absolutely every set, including its Zermelo-Russell set, leads to a contradiction, via much the same adaptation of Russell’s paradox that Zermelo (1908) uses to show that there is no universal set (to distinguish Zermelo’s argument from Russell’s paradox proper, we’ll call it the Russell reductio). Dummett’s case against absolutism then centres on the indefinite extensibility of seemingly comprehensive

\(^1\) By ‘quantification’ I henceforth always mean classical quantification.
concepts, such as object or thing. (1983, chs. 14-16; 1991, p. ch. 24; 1994)

Relativism—the view opposing absolutism—and especially the Dummettian thought that quantification over an absolutely comprehensive domain makes some sort of logical or mathematical mistake, are trenchantly opposed by George Boolos (1993) and Richard Cartwright (1994). The orthodox absolutist view is that the paradoxes neither reveal concepts like set to be indefinitely extensible, nor threaten quantification over absolutely everything. Instead, the straightforward response to the paradoxes is to give up on the naïve-seeming plenitude principles that seem to drive indefinite extensibility. After all, as Cartwright emphasizes (1994, p. 10), it is a logical truth that there is no set whose elements comprise every non-self-membered set (and nothing else).2

My view is that neither absolutism nor relativism is incoherent. The choice between the two views must be made on other grounds. One important consideration is what we might describe as a trade off between ‘generality’ and ‘collectability’. The absolutist makes a prima facie gain in generality. Set theory provides an immediate example. To allow for applications outside pure mathematics, the quantifiers of set theory must range over urelements (non-sets) in addition to sets. The standard impure theory, Zermelo-Fraenkel set theory with Choice and Urelements (ZFCU), is formulated in a language with the set predicate (in symbols: $\text{Set}$) in addition to the usual membership predicate ($\in$). When formulated in this language, its unrestricted quantifiers ($\forall x, \exists y, \forall z$, etc.) seem to cry out for absolute generality. Restricted quantification over sets, when needed, is captured by relativizing unrestricted quantifiers to the set predicate ($\forall s, \exists s$, etc., are henceforth reserved for such restricted quantifiers).3 Suppose, for instance, a set theorist utters the English version of a theorem of ZFCU (or its formalization in the language of ZFCU):

**Singletons** Everything is the sole element of its singleton set.  
$\forall x \exists! z (z \in s \leftrightarrow z = x)$

If we interpret ‘everything’ (or $\forall x$) as ranging over a less than absolutely comprehensive domain, the set theorist fails to rule out singletonless items outside the domain. To capture the intended generality of the theorem seems to call, on the contrary, for quantification over absolutely everything.

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2 We henceforth use ‘comprise’ in this exhaustive sense: zero of more items comprise every $F$ iff every $F$ and nothing else is one of them.

3 $\forall \varphi(s)$ and $\exists \varphi(s)$ abbreviate $\forall \nu(\text{Set}(\nu) \rightarrow \varphi(\nu))$ and $\exists \nu(\text{Set}(\nu) \land \varphi(\nu))$ with $\nu$ chosen to avoid clashes of variable.
On the other hand, the relativist makes a prima facie gain in collectability. Let us immediately disavow any suggestion that collectability is tied to any capacity of physical or mental aggregation we, or our idealized versions, possess. To say that some items are *collected* (as a set), in the relevant sense, is simply to say that some set has exactly those items as its elements. With that clarification made, it seems that a liberal attitude towards collectability is important for applications of set theory. For example, on the model-theoretic approach to semantics, semantic values are encoded using sets. For instance, ignoring intensional features of meaning, the intended extension of a predicate is standardly encoded as the set of items that satisfy the predicate (e.g. the intended extension of ‘donkey’ is the set of donkeys). It’s a well-known theorem of ZFCU, however, that no set has every set as an element. If we suppose that the set theorist is quantifying over an absolutely comprehensive domain, the items that would comprise the extension of ‘set’ are uncollectable. In order to capture the intended extension of the set-predicate in the standard way seems to call instead for us to drop the assumption that the object language is absolutely general, and to frame our semantic theory in a metalanguage quantifying over a more comprehensive domain.

Both sides may attempt to ameliorate their losses. The relativist may seek to compensate for non-absolutely-general quantifiers by seeking non-quantificational ways to generalize (e.g. Shaughan Lavine (2006) deploys schemas, Kit Fine (2006) a suitably interpreted modal operator). Similarly, the absolutist may seek to make up for limited set-collectability by appealing to other non-set-based means to ‘collect’ items (e.g. Agustín Rayo and Gabriel Uzquiano (1999) employ plural resources, Timothy Williamson (2003) higher-order ones). In my view, the debate over the costs and benefits of the resulting packages is far from settled. But settling the trade-off between generality and collectability is much more than can be achieved in a single paper. My more modest aim is simply to reach it.

The main obstacle that lies in the way is a heterodox absolutist response to Russell’s paradox. Williamson (1998) and, more recently, Uzquiano (2015) have suggested absolutist-friendly accounts of indefinite extensibility: a non ad hoc resolution of Russell’s paradox motivates the indefinite extensibility of *set*, but does not require the indefinite extensibility of *object*. The view aspires to combine the best features of both orthodox absolutism and relativism, avoiding the need to trade off generality against collectability. Section IV argues on the contrary that such ‘third way’ absolutism, as we’ll call it, fares poorly on both measures; most damagingly, the view conflicts with (modest subsystems of) ZFCU.

Before we come to that, some preliminary work needs to be done to address a cluster of long standing doubts about the coherence of the case for relativism from indefinite extensibility. Section II outlines some
of these difficulties and Section III attempts to regiment the argument from indefinite extensibility in a way that avoids them.

II.

Relativists are often less than explicit in formulating the plenitude principles they take to drive indefinite extensibility. One of Dummett’s presentations of his argument against absolutism, however, does offer us such a principle:

If there is some determinate totality over which the variable ‘x’ ranges, and if ‘\(F(\xi)\)’ is any specific predicate which is well defined over that totality, then of course there will be some definite subset of objects of the totality that satisfy the predicate ‘\(F(\xi)\)’ … What there is no warrant for is the assumption that the objects so denoted must belong to the totality with which we started. (1981, p. 530)

The key assumption here seems to be a Separation-like principle which we may regiment as follows:

**Totality Separation** Given a predicate \(\varphi(x)\), and any determinate totality, there is a set comprising the members of the totality that satisfy \(\varphi(x)\).

Moreover, with just Totality Separation as a premiss, it’s clear how to employ the Russell reductio to obtain the following conclusion:

**No Comprehensive Totality** No determinate totality comprises everything.

For future reference let’s label this regimentation of the argument, the Dummettian argument. What should the absolutist make of it?

To answer this question requires us to touch on the vexed issue of what Dummett means by terms like ‘determinate totality’ and ‘definite totality’. I should stress however that my primary concern in this section is to draw out some of the difficulties in articulating the case for relativism, rather than offering anything like a comprehensive exegesis of Dummett’s nuanced views about generality.4

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4 Dummett (e.g. 1991, p. 316) sometimes glosses definite totalities in terms of bivalence rather than classical semantics. But, as Boolos (1993, n. 4) observes, this gloss is problematic. See also Oliver (1998, pp. 36–7), who takes up the exegetical challenge eschewed here.
Dummett maintains that the assumption that a quantifier may be interpreted according to its standard (classical) semantics ‘demands that the conception of the domain be completely definite’ (1991, p. 314). His use of ‘totality’-talk leads Boolos to suspect that he takes quantification to require a domain that is a set-like object:

Dummett knows perfectly well that there is ... no set containing all sets, and no class containing all classes. Nevertheless, it would seem he does think that there has to be a—what to call it—totality? collection? domain? containing all the things we take ourselves at any one time to be talking about. He would seem to believe that whenever there are some things under discussion, being talked about, or being quantifier over,... there is a set-like item, a ‘totality’, to which they all belong. (1993, p. 216)

Suppose, then, that we take a determinate totality simply to be a set. In this case, the Dummettian argument emerges, in essentials, as Zermelo’s argument from the Separation Schema in ZFCU, for the conclusion that no set has everything as an element.

Understood in this way, a typical absolutist is likely to take the conclusion of the Dummettian argument to be as harmless as its premiss is uncontroversial. For the lack of a universal set to make trouble for absolutism we would need to draw on further assumptions, such as a strong version of what Cartwright dubs the ‘All-in-One Principle’, the principle that ‘to quantify over certain objects is to presuppose that those objects constitute a “collection,” or a “completed collection”—some one thing of which those objects are the members’ (1994, p. 7). But, as Cartwright emphasizes, there seems little reason to endorse the All-in-One Principle: in order to quantify over some things, it would seem to be enough that there be those things, severally; no additional object—no set-domain—is required. Similar remarks apply if we take definite totalities to be some other kind of set-like object. (1994, p. 8)

Rejecting the All-in-One Principle leaves a residual issue for the absolutist that’s worth addressing before we go any further. How can he best understand ubiquitous ‘domain’-talk, especially in the crucial cases—such as: ‘Some domain is comprehensive’—when he takes there to be no set-like domain comprising the items quantified over. Cartwright, in effect, proposes that the absolutist deploy a no-domain theory of domains akin to the no-class theory of classes: ‘domain’-talk is understood as elliptical for a plural paraphrase, dispensing with set-like domains in favour of their zero or more members, severally (1994, p. 3).

Let’s join Cartwright in rejecting the All-in-One Principle and endorsing the no-domain theory. To implement his suggestion formally, we may enrich the first-order language of ZFCU with plural quantifiers (\(\forall x, \forall y, \ldots\)), informally glossed ‘any zero or more items’ and the
‘member–plurality’ predicate (written: \(u < v v\)), glossed ‘\(u\) is one of the items \(v v\).’ With the help of plural resources, Uzquiano (2009) suggests the following paraphrase of the absolutist’s thesis that some domain is comprehensive:

**Comprehensive Domain** Zero or more things comprise everything.
\[\exists x \forall x (x < xx)\]

Of course, for the no-domain theory to live up to its name, the plural resources need to be taken seriously. The absolutist cannot hope to use plurals to paraphrase commitment to set-like domains out of ‘domain’-talk, unless he rejects the Quinean view that plural quantifiers are bad notation for quantification over set-like items.

To briefly take stock: granted the no-domain theory of domains, Comprehensive Domain, is clearly compatible with the conclusion of the Dummettian argument, No Comprehensive Totality, assuming we take a determinate totality to be a set-like object.

Dummett, however, disavows this construal of determinate totalities. In response to Boolos, he is at pains to avoid the All-in-One assumption, eschewing the use of ‘domain’ or ‘range’, employed as a noun, to show that ‘nothing hangs on the use of such locutions’ in his case against absolutism (1994, p. 248). His strategy for dispensing with set-like totalities would seem to be the very same strategy Cartwright employs for dispensing with set-like domains: Dummett eliminates ‘domain’- and ‘totality’-talk in favour of plural locutions.

This suggests an alternative interpretation of the Dummettian argument. The proposal—to suggestively misspeak—is to identify determinate totalities with *pluralities*. Less loosely, the suggestion is to use plural resources to paraphrase away the ‘totality’-talk in the premiss of the Dummettian argument:

**Plurality Separation** Given a predicate \(\varphi(x)\), for any zero or more things, there is a set comprising the objects among them that satisfy \(\varphi(x)\).
\[\forall x x \exists s \forall x (x \in s \leftrightarrow x < xx \land \varphi(x))\]

So regimented, the negation of Comprehensive Domain follows from the premiss in the plural logic that adds plural analogues of the usual quantifier axioms and rules to classical logic:

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5 This slight departure from the usual ‘one or more’ or ‘two or more’ interpretations streamlines our later arguments. Burgess and Rosen (1997, pp. 151–5) show that ‘zero or more’ can be defined using either of the others.

6 Like Cartwright’s use of ‘domain’-talk, the present use of ‘plurality’-talk is another, increasingly conventionalized, way of requesting a plural paraphrase.
No Comprehensive Domain No zero or more things comprise everything.
\[ \neg \exists x \forall x (x < xx) \]

What should the absolutist make of the plural version of the Dummettian argument?

The difficulty now is not the weakness of the conclusion but the strength of the premiss. Plural first-order logic (PFO) standardly also includes a comprehension axiom ensuring—to suggestively misspeak once more—that every condition determines a plurality.\(^7\) Less loosely:\(^8\)

Plural Comprehension Given a condition \(\varphi(x)\), zero or more things comprise the satisfiers of \(\varphi(x)\).
\[ \exists x \forall x (x < xx \leftrightarrow \varphi(x)) \]

The addition of Plural Comprehension permits us to derive Comprehensive Domain outright. Consequently, Plurality Separation which permits us to derive its negation, No Comprehensive Domain, is inconsistent in PFO.

Rather than being faced with a knock-down refutation of their view, absolutists may leave this encounter with the Dummettian argument feeling that there is no case to answer. For either determinate totalities are taken to be set-like objects and its conclusion is harmless or ‘totality’-talk is paraphrased away in plural terms and its premiss is inconsistent.

III.

Can the relativist formulate her indefinite-extensibility-based case against absolutism in a way that avoids the problems facing the Dummettian argument? The prospects for such an argument on the dialectical terms accepted in the preceding section may seem hopeless. For we have conceded that Comprehensive Domain is a theorem of plural logic. The relativist might respond, of course, by questioning PFO. But a less drastic response is to question whether Comprehensive Domain captures the absolutist’s contention that some domain comprises absolutely everything. After all, Comprehensive Domain emerges as a trivial truth whatever domain its (singular and plural) quantifiers range over. Imagine, for example, a boatless island-dwelling community whose language is just like English save for the fact that they only quantify over the manifestly non-absolutely-comprehensive domain bounded by the horizon visible from their island. All the same,

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\(^7\) We borrow the label, notation and (partial) axiomatization from Linnebo (2013).

\(^8\) Side condition: \(\varphi(x)\) does not contain \(xx\) free.
Comprehensive Domain is true in their language. Their domain is ‘comprehensive’, in the limited, parochial sense of ‘comprehensive’ they employ: zero or more members of their domain comprise every item in their domain.

Similar considerations apply when Comprehensive Domain is expressed in the present version of English or formalized in the language of plural ZFCU. For the sake of simplicity, let’s focus on the latter. The plural language permits us to give a finite plural axiomatization of ZFCU, replacing the non-logical schemas in the first-order theory with single plural axioms. The language is standardly interpreted by specifying a non-empty domain \( M \) for its singular and plural quantifiers, an extensions \( S \) for the set predicate (comprising zero or more members of \( M \)) and an extension \( E \) for the membership predicate (comprising zero or more pairs of members of \( M \)). We assume ‘full’ semantics for plural quantifiers, so that \( \forall x x \) ranges over any zero or more items in \( M \). But even granted this assumption, Zermelo’s (1930) quasi-categoricity theorem demonstrates that plural ZFCU admits of an open-ended sequence of interpretations. Its axioms come out true on any interpretation \( (M,S,E) \) where, \( M \) comprises the cumulative hierarchy of sets and urelements of rank less than some inaccessible cardinal, \( S \) comprises the sets in \( M \) and \( E \) the element-set pairs in \( M \). Rather than single out any one interpretation as \( \text{the} \) intended model of the non-categorical plural theory, its open to the relativist to maintain that ZFCU admits of more and more inclusive intended interpretations, with ever greater inaccessible height. Nonetheless, even if we accept this open-ended Zermelian picture of the set-theoretic hierarchy, the formalization of Comprehensive Domain comes out true under each non-comprehensive interpretation.

By the relativist’s lights, then, the trivial truth of Comprehensive Domain is no threat to relativism. But it does raise a different issue: granted that Comprehensive Domain is true under each intended interpretation of ZFCU, how is the relativist to articulate the conclusion of her anti-absolutist argument? Suppose, for instance, that the language of ZFCU is interpreted by \( (M_0,S_0,E_0) \), and the absolutist and relativist disagree over whether \( M_0 \) is absolutely comprehensive. (The former therefore will refrain from encoding this interpretation as a set-structure, but may instead encode it plurally, following Rayo and Uzquiano (1999).) In order to deny the comprehensiveness of \( M_0 \) in this language there seems to be no better candidate than the trivially false thesis, No Comprehensive Domain.

The relativist may rejoin that this indicates a limitation of the language, so interpreted, rather than a problem for relativism. She may instead seek a more expressive language in which to deny the

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9 This is an instance of a wider statement problem facing relativism. See Williamson (2003) for discussion.
comprehensiveness of $M_0$. Let’s suppose then that, in conjunction with the rest of her linguistic community if need be, she does whatever it is she thinks it takes in order to shift to a more inclusive interpretation of ZFCU. Label the new interpretation $(M_1, S_1, E_1)$. To keep track of this attempted shift in the object language, it’s helpful to add sort indices to the variables and non-logical predicates of the plural language of ZFCU. Sort 0 expressions are interpreted according to the initial interpretation, $(M_0, S_0, E_0)$; sort 1 expressions according to the potentially more liberal one, $(M_1, S_1, E_1)$: this is to say, for $i = 0$ or $1$, singular sort $i$ variables $(x_i, y_i, \ldots)$ and plural sort $i$ variables $(xx_i, yy_i, \ldots)$ range over $M_i$, and the extensions of $Set_i$ and $\epsilon_i$ are respectively $S_i$ and $E_i$. Informally, we shall refer to members of $S_0$ as sets, members of $M_1$ as things, and so on. The syntax remains standard for a plural first-order language, with no ban on cross-sort predications (e.g. both $Set_0 x_0$ and $Set_0 x_1$ are well-formed).

Equipped with the sorted language, the relativist may straightforwardly deny the comprehensiveness of $M_0$:

**No Comprehensiveness Domain**

No zero or more items $\epsilon_0$ comprise every item $\epsilon_1$. $\neg \exists x_0 \forall x_1 (x_1 < xx_0)$

Unlike its unsorted analogue, the truth or falsity of the sorted thesis is no trivial matter. No Comprehensiveness Domain says in effect that neither $M_0$ nor any of its (plurality-encoded) subdomains comprises every member of $M_1$. The sorted thesis comes out true if the relativist succeeds in reaching a more inclusive domain; and it comes out false if she fails.

The sorted language consequently provides a simple way for the relativist to deny the comprehensiveness of $M_0$. It also provides a suitable setting in which to regiment an argument from indefinite extensibility for this conclusion. The first task is to frame a suitable sorted plural logic. The logic must not, of course, prejudge whether the relativist succeeds. But it is technically convenient to suppose that her attempt does not backfire and shrink the domain. We shall consequently suppose—as the absolutist can happily allow—that $M_1$ is no less inclusive than $M_0$. The sorted semantics then sustains a natural generalization of plural logic: suitably sorted, the axioms of PFO come out true and its rules truth-preserving under any pair of structures $(M_0, S_0, E_0)$ and $(M_1, S_1, E_1)$ that meet the condition that $M_0 \subseteq M_1$. Such a pair also renders true a further axiom with no analogue in the unsorted system:

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10 The sorted system includes all instances of the PFO-axioms and rule-schemas in the sorted language subject to two further side-conditions: (i) the plural and singular variable explicit in the Plural Comprehension schema have the same sort (with no constraints on any other variables that occur in the condition); (ii) the sort of the specified variable in the elimination rule for the universal quantifier is no higher than the sort of the universally quantified variable it replaces.
Auxiliary Truism\textsubscript{0,1} Any one\textsubscript{1} of zero or more items\textsubscript{0} is an item\textsubscript{0}.
\[
\forall y_1 (\exists x_0 (y_1 < xx_0 \rightarrow \exists x_0 (y_1 = x_0))
\]

Call the sorted plural logic that adds this axiom to the sorted analogues of the PFO axioms and rules PFO\textsubscript{0,1}

The logic PFO\textsubscript{0,1} neither proves nor refutes No Comprehensive\textsubscript{1} Domain\textsubscript{0}. To argue for this conclusion, the relativist needs further premisses. The argument we consider departs somewhat from the Dummettian argument and has two premisses. The first premiss gives partial voice to the relativist’s maximally liberal attitude towards collectability:\textsuperscript{11}

**Sets\textsubscript{0} get Collected\textsubscript{1}** Any zero or more sets\textsubscript{0} are the elements\textsubscript{1} of a set\textsubscript{1}.
\[
\forall ss_0 \exists s_1 \forall z_1 (z_1 \in_1 s_1 \leftrightarrow z_1 < ss_0) \equiv
\]

This premiss can helpfully be thought of as placing a constraint on the relativist’s attempt at domain expansion. The truth of this premiss requires—to speak loosely—that, in shifting from \( \langle M_0, S_0, E_0 \rangle \) to \( \langle M_1, S_1, E_1 \rangle \), the relativist succeeds in collecting each plurality of sets\textsubscript{0} available in \( M_0 \) into a single set\textsubscript{1} in \( M_1 \).

The second premiss concerns urelements. As usual, urelements are objects that are not sets; similarly, urelements, are objects, that are not sets. The premiss makes a further claim about the relativist’s attempted shift of interpretation:

**Urelements\textsubscript{0} remain Urelements\textsubscript{1}** Every urelement\textsubscript{0} is a urelement\textsubscript{1}.
\[
\forall x_0 (\neg \text{Set}_0 x_0 \rightarrow \neg \text{Set}_1 x_0)
\]

In other words, the premiss requires—speaking loosely\textsuperscript{12}—that shifting from \( \langle M_0, S_0, E_0 \rangle \) to \( \langle M_1, S_1, E_1 \rangle \) never results in non-sets\textsubscript{0} ‘becoming’ sets\textsubscript{1}. For example, we are unable to ‘form’ the Russell set\textsubscript{1} of all sets\textsubscript{0} that lack themselves as elements\textsubscript{0} by fiat, simply by stipulating that Bertrand Russell (the man himself) is in the extension \( S_1 \), and that for any item \( s \), a pair of the form \( \langle s, \text{Russell} \rangle \) is in the extension \( E_1 \) just in case \( s \) is a non-self-membered\textsubscript{0} set\textsubscript{0}. Granted that Russell is a urelement\textsubscript{0}, Urelements\textsubscript{0} remain Urelements\textsubscript{1} rules out his being made a set\textsubscript{1} in this way.

The conclusion, No Comprehensive\textsubscript{1} Domain\textsubscript{0} then follows from the two premisses, Sets\textsubscript{0} get Collected\textsubscript{1} and Urelements\textsubscript{0} remain Urelements\textsubscript{1} in PFO\textsubscript{0,1}, via a sorted plural version of the Russell reductio.

\[\text{\textsuperscript{11} } \forall ss_0 \varphi (ss_0) \text{ abbreviates } \forall vv_0 (\forall v_0 (v_0 < vv_0 \rightarrow \text{Set}(v_0)) \rightarrow \varphi(vv_0)) \text{ with } vv_0 \text{ chosen to avoid clashes of variable.}\]

\[\text{\textsuperscript{12} } \text{Much like ‘collecting’-talk, few relativists take the process-metaphor in ‘becoming’ and similar terms literally.}\]
Proof sketch. We reason informally in PFO, 1. Suppose Comprehensive Domain, for reductio. Applying Sets get Collected (and Plural Comprehension) we obtain a Russell set, \( \{ x_0 : \text{Set}_0 x_0 \land x_0 \in x_1 x_0 \} \) that has as elements those sets, that lack themselves as elements. By Zermelo’s tweak of Russell’s reasoning, the Russell set is not a set. But nor is it a urelement, by Urelements remain Urelements. It follows that the Russell set is not a thing. But it is one of the zero of more things that Comprehensive Domain says comprise every thing. This contradicts Auxiliary Truism.

What should the absolutist make of the argument? Assuming he does not object to reasoning in PFO, 1, he must either accept the conclusion or reject a premiss. The former option is, strictly speaking, compatible with absolutism. For the relativist’s conclusion only denies the comprehensiveness of a single domain, \( M_0 \). The absolutist might go back on his initial claim that \( M_0 \) is absolutely comprehensive without renouncing absolutism.

The dialectical position this leaves him in, however, is clearly untenable.\(^{13}\) For the argument made no special assumptions about the initial interpretation \( \langle M_0, S_0, E_0 \rangle \). For the absolutist to claim, on the strength of the relativist’s argument, that \( M_0 \) fails to be absolutely comprehensive, after all, but that it is instead \( \langle M_1, S_1, E_1 \rangle \), or some other interpretation, that really has an absolutely comprehensive domain simply invites the relativist to repeat her attempted domain shift and run the argument again. Indeed we already know exactly the form the argument is going to take: it’s an instance of the argument schema that results from replacing the sort indices 0 and 1 with schematic variables \( i \) and \( j \).\(^{14}\) Once the absolutist has conceded that the instances of Sets, get Collected, and Urelements, remain Urelements, suffice to establish the corresponding instance of No Comprehensive Domain, in PFO, in the case when \( i \) and \( j \) are respectively replaced by 0 and 1, what grounds could he have to resist coming to the same conclusion in the case of any other instance?

The other option seems less unappealing. The absolutist can simply reject one of the relativist’s premisses. The obvious premiss to choose is Sets get Collected, or, if he takes a little while to find his resolve, some other instance of Sets, get Collected. The premiss flows from the Zermellian conception, described above, where plural ZFCU admits of an open-ended sequence of intended models with wider and wider domains:

\[ \langle M_0, S_0, E_0 \rangle \langle M_1, S_1, E_1 \rangle \langle M_2, S_2, E_2 \rangle \ldots \]

\(^{13}\) Compare Williamson (2003, pp. 434–5).

\(^{14}\) Side condition (on this, and all subsequent schemas containing \( i \) and \( j \)): the sort index replacing \( j \) labels an interpretation that results from the relativist’s attempt to suitably expand the domain of the interpretation labelled by the sort index replacing \( i \).
An instance of Sets, get Collected, comes out true when interpreted by any pair of interpretations in the sequence \( (M_i, S_i, E_i) \) and \( (M_j, S_j, E_j) \), with \( i < j \).

But absolutists typically reject this conception of the cumulative hierarchy in favour of something closer to Cantor’s conception of the set-theoretic universe. Of course, assuming he accepts modest Large Cardinal axioms (e.g. there is an unbounded sequence of inaccessible cardinals), he is committed to there being an unbounded sequence of set-models of plural ZFCU of the kind Zermelo describes. Nonetheless, the absolutist may claim, no set-model encodes an \textit{intended} interpretation of ZFCU. Instead, on this account, the sole (non-set-encoded) intended interpretation—label it: \( (M_\omega, S_\omega, E_\omega) \)—has a domain \( M_\omega \) that comprises absolutely everything (i.e. every thing, thing, ...) and extensions \( S_\omega \) and \( E_\omega \) that respectively comprise absolutely every set (i.e. every set, set, ...) and absolutely every element–set pair. On this account, assuming she starts with the intended interpretation, the relativist’s iterated attempts to expand the universe fail. The best she can hope for is to leave the interpretation static, so that each model in the resulting sequence is the maximal one: \( (M_i, S_i, E_i) = (M_\omega, S_\omega, E_\omega) \). So interpreted, each instance of Sets, get Collected, comes out false.

The choice between the Zermelian and Cantorian conceptions of the set-theoretic hierarchy brings us back to the trade off between generality and collectability mentioned in Section I. In taking ZFCU to have a single intended interpretation \( (M_\omega, S_\omega, E_\omega) \), with an absolutely comprehensive domain, the absolutist secures what seems to be the intended generality of set theory. But the price he pays for this means to achieve absolute generality is the loss of unlimited collectability. In particular, he cannot follow the relativist in maintaining that it’s always open to us to encode the intended extensions of our predicates as set-extensions in the context of a suitably liberal metalanguage.

\textcolor{red}{IV.}

There is, however, a third way to respond to the argument.

Dummett sometimes seems to assume that a concept with an indefinitely extensible sub-concept is automatically itself indefinitely extensible. He explains that quantification over absolutely every object is impossible because ‘the (formal) concept object (or identical with itself) embraces all others; since some of the others are indefinitely extensible, it must be also’ (1994, p. 249). But this assumption is contestable. Recent work by some philosophers who endorse absolutism, combines
this view with something much like the indefinite extensibility Dummett attaches to set.

Williamson suggests that attaching wider and wider extensions to the set predicate permits absolutists to accommodate the intuitions behind indefinite-extensibility-based arguments for relativism:

For given any reasonable assignment of meaning to the word ‘set’ we can assign a more inclusive meaning while feeling that we are going on the same way ... The inconsistency is not in any one meaning ... it is in the attempt to combine all the different meanings that we could reasonably assign it into a single super-meaning. (1998, p. 20)

In a similar vein, Uzquiano seeks to reconcile absolute generality with a ‘linguistic’ account of the indefinite extensibility of ‘set’. Citing Gödel’s presentation of the iterative conception of set according to which ranks of sets are iteratively added on top of some antecedently given domain of urelements, his aim is to ‘reframe Gödel’s procedure in terms of a cumulative process of reinterpretation of the primitive set-theoretic vocabulary’, deploying modal resources to describe the possible reinterpretations (2015, p. 150). Again, on his view, the ever more liberal extensions for the set predicate are always contained within the absolutely comprehensive domain.

These accounts suggest a different response to the relativist’s argument from Section III. Something like iterated relativist-style attempts to liberalize the interpretation do results in a sequence of different interpretations for ZFCU:

\[ (M_\infty, S_0, E_0), (M_\infty, S_1, E_1), (M_\infty, S_2, E_2), ... \]

Only, unlike on the relativist’s Zermellian picture, the more and more liberal extensions for the set- and membership-predicate lie within the comprehensive domain: \( S_0 \subset S_1 \subset \cdots M_\infty \) and \( E_0 \subset E_1 \subset \cdots M_\infty \). This permits the third-way absolutist, as we’ll call him, to combine aspects of both orthodox absolutism and relativism: he maintains, we suppose, that the sequence of intended interpretations renders true each instance of \{sets, get Collected, and Comprehensive, Domain\}. He instead rejects the second premiss, Urelements, remain Urelements; on this view, members of the absolutely comprehensive domain that are not sets, (i.e. urelements,) may ‘become’ sets, relative to more liberal interpretations.

On the face of it, third way absolutism may seem well placed to avoid trading off generality against collectability. For like orthodox absolutism, this view permits quantification over an absolutely comprehensive domain. Moreover, in common with relativism, this view upholds the maximally liberal attitude towards which pluralities of sets...
are collectable. When we consider wider issues concerning generality and collectability, however, the third way compromise fares poorly.

Take generality first. In order to accord ZFCU the generality it seems to strive for the orthodox absolutist may draw on quantification over an absolutely comprehensive domain. But this alone is not enough. We also require a means to capture restricted quantification over a domain comprising absolutely every set. Standardly, this is achieved by relativizing the quantifier to the set predicate. Consider, for instance, the ZFCU formulation of the Power Set axiom (with the relativization made explicit):

**Power Set** Every set has a power set.
\[ \forall x (\text{Set} x \to \exists y (\text{Set} y \land \forall z (z \in y \leftrightarrow z \subseteq x))) \]

According to third way absolutism, however, any intended interpretation of Power Set is given by a structure of the form \( \langle M, S_i, E_i \rangle \). Although \( \forall x \) ranges over the absolutely comprehensive domain, \( \text{Set} \) is interpreted by an extension that fails to comprise absolutely every set (i.e. every \( s_{i, n} \), \( s_{i, 1}, \ldots \)). So interpreted, Power Set fails to rule out power-set-less sets, outside \( S \). So far as framing a maximally general axiomatization of ZFCU goes, third way absolutism offers little improvement on relativism.

The temptation at this stage may be to try to combine all the intended extensions for \( \text{Set} \) and \( \in \) into single super-extensions, \( S_{\omega} \) and \( E_{\omega} \). The putative super-extension \( S_{\omega} \) comprises everything in \( S_\xi \), everything in \( S_\eta \), and so on. Of course, by the absolutist’s lights, every object that the putative super-extension \( S_{\omega} \) would contain is available in the absolutely comprehensive domain. The question remains whether the absolutist can lasso them into a single (non-set-encoded) extension which he takes to capture the intended meaning of the set predicate. Similarly, *mutatis mutandis*, for \( E_{\omega} \).

Here the third way absolutist faces a dilemma. One option is to follow Williamson in rejecting such super-meanings. The cost is that we must forego quantification over a domain comprising absolutely every set. The availability of a quantifier \( \exists s \) with this domain goes hand in hand with the availability of the super-meaning for the set predicate: for we may explicitly define \( \text{Set} x \) as \( \exists s(s = x) \).

The alternative is to allow for super-predicates \( \text{Set}_{\omega} \) and \( \in_{\omega} \) that do express the super-meanings. Uzquiano (2015, p. 155), in effect, takes this route, by endorsing modal resources that permit him to define the super-predicates. But such a move simply invites the relativist to re-run her argument, treating the \( \omega \)-subscript as just another sort index. The

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15 The super-meaning is captured by the ‘modalization’ of the set predicate: \( \ast \text{Set} x \)
Collectability raises further problems for third way absolutists. So far we have focused on the question of which pluralities of sets are collectable. In this case, third way absolutists join relativists in sustaining a liberal attitude, endorsing each instance of $S$, get Collected. But what about arbitrary pluralities, including pluralities of urelements and pluralities of both sets and urelements? The relativist can coherently extend her liberal attitude across the board, accepting each instance of the following:

**Things, get Collected**. Any zero or more things, are the elements, of a set, $S$.

$$\forall x \exists y \forall z_j (z_j \in y \leftrightarrow z_j < xx)$$

But the absolutist cannot follow suit. With a further tweak to the Russell reductio, Things, get Collected, proves No Comprehensive, Domain, in PFO, without further assumptions.

This raises the question of *which* pluralities are uncollectable. Stewart Shapiro (2003) notes one important example of a plurality that is uncollectable by the lights of third way absolutism. The theory ZFCU leaves open whether or not there is a set comprising every urelement. Vann McGee (1997) suggests settling the question positively, with the addition of the Urelement Set Axiom. In the present context, the axiom emerges as a schema:

**Urelement, Set, Axiom**. Some set, has every urelement, as an element, $s_i$.

$$\exists x \forall i (\neg \exists s_i x_i \rightarrow x_i \in s_i)$$

But, as Shapiro observes on the basis of cardinality considerations, assuming the package of views endorsed by the third way absolutist, this axiom ‘fails and fails badly in any language which has a set theory like ZFC and a quantifier ranging over absolutely everything’ (2003, p. 477). His observation is straightforwardly regimented in PFO, $\forall i$. Write ZFCU, for the notational variant of plural ZFCU that uniformly indexes all variables and non-logical predicates with the index $i$. Then in conjunction with ZFCU, and ZFCU, the characteristic theses of the third
way absolutist, Sets, get Collected, and Comprehensive, Domain, refute the Urelement, Set, Axiom.

Is this a problem for third way absolutism? Shapiro doesn’t take it to be a conclusive objection (2003, p. 478). The limits third way absolutism imposes on collectability, however, run rather deeper. ZFCU, as we just saw, is sometimes silent on whether or not a given plurality is collected; but very often it is not. Consider the following theses about the collectedness of arbitrary pluralities:

C1: Any plurality of at most set, many objects, is collected by a unique set.

C2: Any plurality of at most countable many objects, is collected by a unique set.

C3: Any plurality of at most $n$ objects, is collected by a unique set, (for fixed finite $n$).

C4: Any plurality of at most two objects, is collected by a unique set.

Each collectability thesis $C1, \ldots, C4$, is a theorem of plural ZFCU; $C2, \ldots, C4$, are also theorems of its subtheory which restricts the axiom of Replacement, to countable sets; each instances of $C3$, (for fixed finite $n$) is also a theorem of the subtheory which only contains four of its first-order axioms: Extensionality, Empty Set, Pairing, and Union; finally, $C4$, is a theorem of the subtheory comprising just Extensionality, Empty Set, and Pairing. Nonetheless, the third way absolutist must reject each of $C1, \neg C4$.

The trouble again concerns cardinality: the third way absolutist package of Comprehensive, Domain, and Sets, get Collected, refutes $C4$, in PFO.$\omega$.

Proof sketch. Suppose for reductio that $C4$ holds. Then assuming Comprehensive, Domain, each set, (which is also an object) is associated with a unique set, namely its singleton. This goes to show that sets, are at least as numerous as sets. On the other hand, Sets, get Collected, implies that there are more sets, than there are sets, (by Cantor’s diagonal argument).

The same package of assumptions consequently also refutes $C1$, $C2$, and $C3$, in PFO.$\omega$ (assuming in the first case that there is at least one two-membered set).

The price we pay for third way absolutism, then, is to place severe limits on which pluralities are collectable. Not only must the third way absolutist forego extending ZFCU with the Urelement Set Axiom, he
must give up on ZFCU itself. Indeed, even the meagre combination of Extensionality, Empty Set, and Pairing is off limits for third way absolutism.

If the standard way to formulate set theory with urelements is unavailable to the third way absolutist, might he instead appeal to a non-standard theory? He may object that ZFCU, fails to respect an important distinction between ‘permanent’ urelements, such as donkeys and spacetime points that never enter the extension of the set predicate, and ‘temporary’ urelements, such as the extension S, itself that do eventually ‘become’ sets, when we reach a sufficiently liberal structure.

In much this spirit, Uzquiano (2015, p. 155) suggests interpreting set theorists’ assertions to be tacitly restricted to a domain of available objects. Restricted Pairing may then be formulated, by adding a further unary predicate \( Av \) (read ‘available’) to the language of ZFCU:

\[
\forall x \forall y (Avx \land Avy \rightarrow \exists s \forall z (z \in s \leftrightarrow z = x \lor z = y))
\]

**Restricted Pairing** Any one or two available objects comprise the elements of a set.

The other axioms may be restricted to available items in a similar way.

Setting aside the obvious loss of generality, the main difficulty with Restricted Pairing is that it curtails the applicability of pair sets. One important application of pairing is to define ordered pairs and \( n \)-tuples as sets in the standard way due to Kuratowski. With unrestricted Pairing, the orthodox absolutist obtains an \( n \)-tuple \( \langle a_1, \ldots, a_n \rangle \) for any items \( a_1, \ldots, a_n \). This permits him to encode arbitrary extensions as pluralities of \( n \)-tuples; for example, the intended extension of the identity predicate is the plurality-extension comprising the ordered-pairs \( \langle a, a \rangle \) whose co-ordinates are identical. But by restricting Pairing to available objects, the third way absolutist forgoes Kuratowski \( n \)-tuples of unavailable objects, losing this non-set-based means to encode arbitrary extensions.

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\(^{16}\) Uzquiano’s proposal centres on the ‘modalization’ of Restricted Pairing, prefixing each non-logical predicate with \( \ast \). But the modal axiom faces the same problem as its non-modal counterpart.
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