On Type Distinctions and Expressivity

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Formal languages: an embarrassment of riches

- There is a wide variety of formal languages.

- One difference is whether the language contains type distinctions.

  - First-order logic: *one style of variables* and matching quantification

    *un*typed language

  - Higher-order logic: *two or more styles of variables* and quantifiers

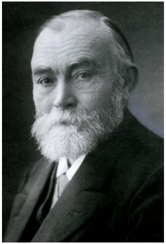
    typed language

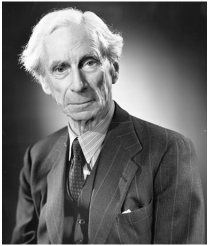
Opposing traditions

Main question

Does a single style of variables suffice for theoretical inquiry?

- Positive answer: tradition famously represented by Quine
- Negative answer: Frege, Russell, ... higher-order metaphysics





Justifying the choice

Justification

What considerations support the adoption of a framework with one style of variables rather than many?

- Some think that richer type-theoretic structures are
  - superfluous;
  - illegitimate;
  - unintelligible.
- Others think that richer structures are
  - theoretically important;
  - indispensable.

Expressivity

Focus: important but puzzling role played by the notion of expressivity

- Typed languages seem more expressive.
- Typed languages are said to have expressive limitations.
- Untyped languages have been criticised for being too expressive.
  - They permit the formalisation of nonsense.
  - They force unnecessary theoretical choices.

My goals

- Develop some of these argumentative strategies
- Show that expressivity does not favour typed languages

(1) Appealing to greater expressive power in support of typed languages is not effective.

(2) Untyped languages are not too expressive.

Partial vindication of untyped languages

Plan

1. Justify the use many-sorted logic as a framework for the debate
2. Argue that typed languages *don't* offer greater expressive power...
    - ... in a semantic sense;
    - ... in a syntactic sense.
3. Argue that untyped languages are not too expressive.

Signature

- Set of sorts

  e.g. $\{\mathrm{o}, \mathrm{p}\}$ or $\mathbb{N} = \{0, 1, 2, 3 \ldots\}$.

- Associated styles of variables and matching quantifiers

  e.g. $\exists x^{\mathrm{o}} \varphi(x^{\mathrm{o}}), \exists x^{\mathrm{p}} \varphi(x^{\mathrm{p}}), \ldots$

  $\exists x^0 \varphi(x^0), \exists x^1 \varphi(x^1), \exists x^2 \varphi(x^2), \ldots$

- Predicates and functions with appropriate sortal restrictions

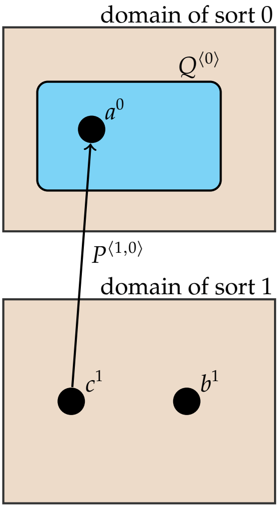
  e.g. $R(\underbrace{\ldots.}_{\text{sort } 0}, \underbrace{\ldots.}_{\text{sort } 1})$ $\qquad R(x^0, x^1)$ $\qquad \cancel{R(x^1, x^0)}$

  $R^{\langle 0,1 \rangle}(\underbrace{\ldots.}_{\text{sort } 0}, \underbrace{\ldots.}_{\text{sort } 1})$

Recovering type-theoretic frameworks

	sorts	examples
first-order logic	$\{0\}$	$P(y) \mapsto P^{\langle 0 \rangle}(y^0)$
second-order logic	$\{0, 1\}$	$X(y) \mapsto P^{\langle 1,\, 0 \rangle}(x^1, y^0)$
strict type theory	$\{0, 1, 2, \ldots\}$	$x^{n+1}(y^n) \mapsto P^{\langle n+1,\, n \rangle}(x^{n+1}, y^n)$
cumulative type theory	$\{0, 1, 2, \ldots\}$	$x^n(y^m) \mapsto P^{\langle n,\, m \rangle}(x^n, y^m)$ $(n > m)$
liberal type theory	$\{0, 1, 2, \ldots\}$	$x^n(y^m) \mapsto P^{\langle n,\, m \rangle}(x^n, y^m)$ (any n, m)

Semantics


domain of sort 0
$Q^{\langle 0 \rangle}$
a^0
$P^{\langle 1,0 \rangle}$
domain of sort 1
c^1
b^1

A case for many-sorted logic

- When comparing typed languages:
  - Debates about the correct form of type theory become debates about which predicates to allow.
  - Predication as one among many fundamental relations.
  - The distinction between ontology and ideology is especially clear.
- When comparing typed and untyped languages:
  - Straightforward translation between many-sorted and one-sorted
  - Potentially useful results concerning theoretical equivalence

Two ways of understanding the question

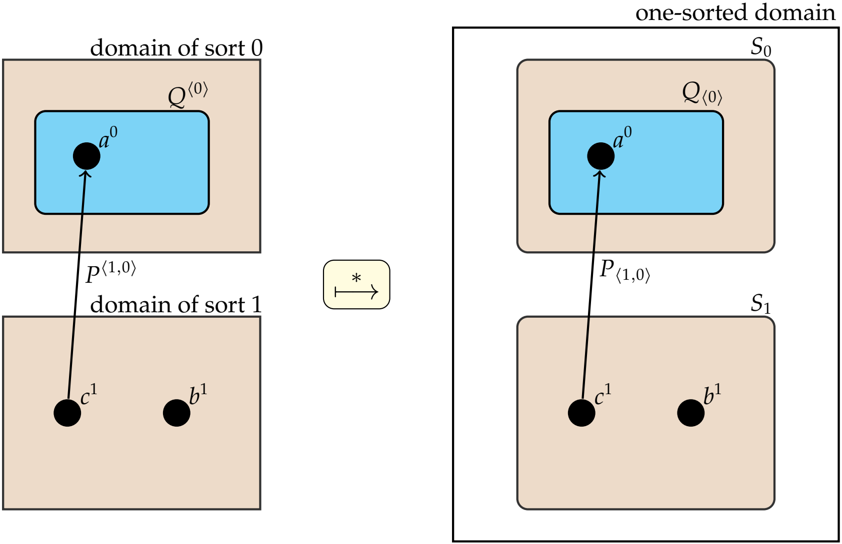
Present question

Do many-sorted systems really have more expressive power than those with one sort only?

(1) *Semantic*: can more be *represented* in many-sorted logic than in one-sorted logic?

(2) *Syntactic*: can more be *said and proved* in many-sorted logic than in one-sorted logic?

Model conversion



Translation from many-sorted to one-sorted

$$R^{\langle i_1,\dots,i_n\rangle}(t^{i_1},\dots,t^{i_n}) \quad \overset{\bullet}{\longmapsto} \quad R_{\langle i_1,\dots,i_n\rangle}(t_{i_1},\dots,t_{i_n})$$

$$t^{i_1} = t^{i_2} \quad \overset{\bullet}{\longmapsto} \quad t_{i_1} = t_{i_2}$$

$$\varphi \wedge \psi \quad \overset{\bullet}{\longmapsto} \quad [\varphi]^\bullet \wedge [\psi]^\bullet$$

$$\neg\varphi \quad \overset{\bullet}{\longmapsto} \quad \neg[\varphi]^\bullet$$

$$\exists x^i \varphi \quad \overset{\bullet}{\longmapsto} \quad \exists x_i (S_i(x_i) \wedge [\varphi]^\bullet)$$

Correspondence

Correspondence Theorem

M satisfies a sentence σ if and only if M^* satisfies $[s]^\bullet$:

$$M \vDash \sigma \;\Leftrightarrow\; M^* \vDash [\sigma]^\bullet$$

- No important difference between M and M^* re expressivity:
  - The models are about same entities.
  - Primitives have the same denotations.
  - Sentences σ and $[\sigma]^\bullet$ have the same truth conditions.

Two semantic perspectives

- I have assumed an untyped metatheory (standard set theory).
- From this perspective, there is no additional expressivity.
- There is a different perspective: *typed metatheory*.
- A popular example is higher-order semantics.
  - Draw possible semantic values from multiple types
  - Common for plural logic or, more recently, higher-order logic
  - Primitive second-level entities interpreting second-order variables

Typed metatheory

- This perspective yields a different verdict.
- Many-sorted models seem richer than one-sorted models.
  - The models are *not* about same entities.
  - Primitives *don't* have the same denotations.
  - Sentences σ and $[\sigma]^\bullet$ *don't* have the same truth conditions.

Standoff

- Standoff between two consistent perspectives
- Assume an *un*typed metatheory:

  many-sorted languages *do not appear* more expressive.
- Assume a typed metatheory:

  many-sorted languages *appear* more expressive.
- Appealing to expressivity has little dialectical weight.

Conditional conclusion

- If we adopt a typed metatheory, then many-sorted languages seem more expressive.
- Positive reasons to adopt a typed metatheory?
- Prominent argument due to Timothy Williamson (2003)
- Making reflective sense of absolute generality requires type distinctions in the metatheory.

Argument

Three assumptions are inconsistent:

(1) Liberal Principle of Interpretations

(2) Absolute generality is possible.

(3) Interpretation are objects.

Liberal Principle of Interpretations

Liberal Principle of Interpretations

Let P be a predicate. For any domain D and for any formula φ, there is an interpretation J according to which P applies to an object in D if and only if the object satisfies φ.

Example. Formula φ: 'is prime'; domain: $\mathbb{N}$.

There is an interpretation according to which P applies exactly to the natural numbers that are prime.

Liberal Principle of Interpretations (unrestricted domain)

For any formula φ, there is an interpretation J according to which P applies to an object if and only if the object satisfies φ.

Contradiction

- Set

  φ = ‘is not an interpretation x according to which P applies to x’

  Contradition follows.

- Williamson rejects (2): interpretations are not objects...

  but primitive second-level entities.

- The argument can be made about an arbitrary sort i.

A more fine-grained analysis

Plenitude of model-theoretic interpretations

For any combination of *suitable semantic values* for the expressions of the object language, there is an interpretation that assigns that combination of semantic values to those expressions.

Suppose the suitable semantic value of predicates are properties.

- We can factor the liberal principle into two components.

(i) For any property, there is an interpretation assigning that property to P as semantic value.

(ii) For any formula φ, there is a property that an object has if and only if the object satisfies φ.

(ii): naive comprehension for properties

- Type distinctions: safe and well understood of approaching (ii)

- Attempts do to justice to (ii) without type distinctions
  - Non-classical (e.g. Field 2004)
  - Classical (e.g. Schindler 2019)

- Reject (ii):
  - Analogy with set theory
  - Conceptions of property
  - Broad theoretical considerations

Upshot

- Type distinctions provide a viable but not inevitable solution.
- We need not assume that properties...
  - satisfy a form of naïve comprehension;
  - this is best captured by postulating a type distinction.
- It's an open debate: no one can plausibly have the last word yet.

Two ways of understanding the question

Present question

Do many-sorted systems really have more expressive power than those with one sort only?

(1) *Semantic*: can more be *represented* in many-sorted logic than in one-sorted logic?

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Trivially true thesis?

- The thesis seems trivially true.
- Any one-sorted system can be properly extended by a many-sorted system.
  - Add new sorts and new predicates connecting them.
  - This yields more validities, owing to the logical axioms of the system.
- This is not necessarily significant.

Equivalence

Semantic Equivalence Theorem

For any many-sorted Σ and σ,

$$\Sigma \models \sigma \;\Leftrightarrow\; \Sigma^{\bullet} \cup \Delta \models [\sigma]^{\bullet}$$

where Δ conveys that each S_i has a witness, i.e. $\Delta = \{\exists x S_i x : i \in I\}$.

Syntactic Equivalence Theorem

For any many-sorted Σ and σ,

$$\Sigma \vdash \sigma \;\Leftrightarrow\; \Sigma^{\bullet} \cup \Delta \vdash [\sigma]^{\bullet}$$

Significance

- Central motivation for the programme of higher-order metaphysics: to regiment metaphysical reasoning.
- Any many-sorted system that faithfully represents reasoning in a metaphysical domain can be matched inside a one-sorted system.
- Typed languages might be heuristically good, even optimal.
- This applies, *mutatis mutandis*, to any other area of inquiry faithfully regimented in a many-sorted system.

Are typed languages too expressive?

- Expressivity plays a puzzling role in this debate.
- Untyped languages are said to overcome expressive limitations of typed languages.
  - See generalisations requiring quantification over types.
- Does the expressivity of untyped languages goes too far?
  - It permits the fomalisation of nonsense.
  - It forces us to make unnecessary theoretical choices.

Category mistakes

- Morphological or syntactical rules of English allow:
  - 'Every prime number is green'.
  - 'Every chair has a prime factorisation'.
- A many-sorted language has the resources to avoid these results.
- A one-sorted language doesn't.

Objection: don't allow this kind of nonsense!

- Strong overall case that category mistakes are in fact meaningful (see Magidor 2013, ch. 3).

  (a) category mistakes are context-sensitive in a way that meaninglessness is not;

  (b) unlike nonsense, category mistakes can be felicitously embedded in attitude reports;

  (c) a compositional theory of meaning will likely imply that many paradigmatic cases of category mistakes have meaning;

  (d) assuming that category mistakes are meaningful best explains their use in figurative speech.

- Category mistakes tend to be practically pointless... but protecting us from futility has never been the goal of a formal system.

Objection: block unnecessary theoretical choices!

- As already noted by Quine (1956), switching from many-sorted system to one-sorted system opens up new theoretical questions.

[E]liminating sort distinctions forces us to make unnecessary conventional choices about how to extend predicates beyond their original range of application. [...] [The many-sorted framework] does not force us to apply predicates in cases where we have no good reason to say that they do (or do not) hold of the items in question (Barrett and Halvorson 2017, 3578).

Response from an axiomatic perspective

- These conventional choices are not always forced.
- We may be happy for our theories to remain incomplete.
- If conventional choices are required, there is no technical obstacle. (Lindenbaum's lemma)

Response from a model-theoretic perspective

- A model will require conventional choices.
- These choices are in the metatheory and only relative.
- They don't settle how the one-sorted theory is extended.

Interesting theoretical choices

- Examples from Quine 1956:
  - Should we identify certain classes with sets?
  - Should we identify empty classes from different types?
  - Should we identify an individual with its singleton set?
- Example from plural logic:
  - Should we identify a single-membered plurality with its member?

Conclusion

- Expressivity does not favour typed languages over untyped ones.
- Appealing to expressive power is not effective.
  - Semantically understood, it delivers a conditional conclusion.
  - Syntactically understood, untyped languages fare just as well as typed languages.
- Untyped languages are not too expressive in problematic ways.

Conclusion

- Typed languages might still be preferable because of other theoretical or practical considerations.

- As far as expressivity is concerned, one may continue to follow Quine's advice.

[...] pool types and get on with homogeneous variables [...]

(Quine and Carnap 1990, p. 353)